which give the ratios of x, y, and z for the 8 cases corresponding to the set of internal bisectors. If we take the positive square roots in (4) and (5), we find, using the plus signs,

$$x : y : z = (1 + u)^{-2} : (1 + v)^{-2} : (1 + w)^{-2} \tag{6}$$

and, using the minus signs,

$$x : y : z = u^2(1 + u)^{-2} : v^2(1 + v)^{-2} : w^2(1 + w)^{-2} \tag{7}$$

where

$$u = \tan A/4, \qquad v = \tan B/4, \qquad w = \tan C/4;$$

and hence

$$1 - \Sigma u - \Sigma uv + uvw = 0. \tag{8}$$

By (1), the actual values of x, y, and z are $\dfrac{(1+v)(1+w)}{2(1+u)}$, etc., in the case of (6) and $\dfrac{u(1+v)(1+w)}{2vw(1+u)}$, etc., in the case of (7). From these two solutions we can get, not only the 8, but all the 32, by replacing A, B, C by $A + l\pi$, $B + m\pi$, $C + n\pi$, where $l + m + n$ is a multiple of 4, since these angles apply equally well to the triangle, and they leave the relation (8) unaltered.

To get a practical construction, let us denote by $\rho_1, \rho_{11}, \rho_{12}, \rho_{13}; \rho_2, \rho_{21}, \rho_{22}, \rho_{23}, \cdots$, the in- and ex-radii of AFI, BDI, CEI. Then

$$\rho_{11} = \frac{1+u}{2}r, \qquad \rho_{12} = \frac{1+u}{2u}r, \text{ etc.}$$

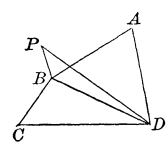
Hence in the case of (6), the radii of the required circles are the three fourth proportionals to ρ_{11}, ρ_{21}, ρ_{31} in different orders; and in the case of (7) the fourth proportionals to ρ_{12}, ρ_{22}, ρ_{32}. The remainder of the set of 8 cases may be solved by using other combinations of the ρ's, while all the 32 cases may be similarly treated by means of the triangles AF_1I_1, etc.

495. Proposed by N. P. PANDYA, Sojitra, India.

A point P moves so that the quadrilateral $PBCD$ is half of a given quadrilateral $ABCD$. Find the locus of P.

SOLUTION BY J. W. BALDWIN, University of Michigan.

In general the triangles ABD and BCD will not be equal. Let ABD be the larger of the two. Then we are to have triangle BCD + triangle PBD equal to half of the given quadrilateral $ABCD$ for all positions of P. That is, the triangle PBD must have a constant area; and having a fixed base BD must have a constant altitude, the distance from P to BD or BD produced. Hence the locus of P is a line parallel to BD. In case triangle ABD = triangle BCD the locus of P is the line of which BD is a segment and two sides PB and PD of the quadrilateral $PBCD$ fall in this line.



Also solved by W. J. THOME, G. W. HARTWELL, WILLIAM HOOVER, S. W. REAVES, W. R. RANSOM, J. W. CLAWSON, and the PROPOSER, some solvers using analytic methods and one using trilinear, perpendicular coördinates.

CALCULUS.

407. Proposed by PAUL CAPRON, Annapolis, Maryland.

A coffee pot in the form of a conical frustum, 10 inches high, with a lower base 8 inches in diameter and an upper base 6 inches in diameter, is held on a slant so that the lower base is barely covered by the coffee within, and the upper base is barely uncovered. How much coffee does the pot contain?

III. SOLUTION BY WILLIAM W. JOHNSON, Cleveland, Ohio.

The quantity of coffee in the pot is equal to the volume of the conical ungula C-$APBQ$ formed by tipping the conical frustum on a slant, according to the conditions of the problem. In the figure let $AL = R$, $DF = r$, $FL = h$, $GO = x$, $FO = y$. Then,

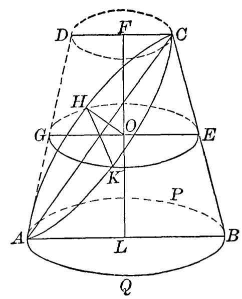
$$\text{Volume } C\text{-}APBQ = V = \int_r^R Sdy = \frac{h}{R-r}\int_r^R \left(x^2 \operatorname{arc\,cos} \frac{2Rr-(R+r)x}{(R-r)x}\right.$$
$$\left. - \frac{1}{(R-r)^2}[2Rr-(R+r)x]\sqrt{4Rr(R+r)x-4Rrx^2-4R^2r^2}\right) dx,$$

in which, area $HKE = S$, and

$$dy = \left(\frac{h}{R-r}\right) dx;$$

since

$$y = \frac{h(x-r)}{R-r}.$$



Integrating, we obtain

$$V = \frac{\pi h R^{\frac{3}{2}}}{3(R-r)}(R^{\frac{3}{2}} - r^{\frac{3}{2}}).^{1}$$

Putting $h = 10$, $R = 4$, and $r = 3$, we get

$$V = \frac{80}{3}(8 - 3\sqrt{3})\pi = 234.895 \text{ cu. in.}$$

409. Proposed by B. J. BROWN, Victor, Colorado.

Integrate the equation

$$\frac{\partial^2 z}{\partial x \partial y} + \frac{1}{x+y}\left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right) - \frac{2}{(x+y)^2} z = 0.$$

SOLUTION BY O. S. ADAMS, Coast and Geodetic Survey, Washington, D. C.

Let

(1) $$z = \frac{u}{(x+y)^2}.$$

Then, computing $\partial z/\partial x$, $\partial z/\partial y$ and $\partial^2 z/\partial x \partial y$, and substituting in the given equation, we may write the result as follows,

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} - \frac{u}{x+y}\right) - \frac{1}{x+y}\left(\frac{\partial u}{\partial x} - \frac{u}{x+y}\right) - \frac{2u}{(x+y)^2} = 0.$$

Now let

(2) $$\frac{\partial u}{\partial x} - \frac{u}{x+y} = v.$$

Then

(3) $$u = \frac{(x+y)^2}{2}\frac{\partial v}{\partial y} - \frac{x+y}{2} v.$$

Computing $\partial u/\partial x$ and $u/(x+y)$, and substituting in (2), we have

$$\frac{\partial u}{\partial x} - \frac{u}{x+y} = \frac{(x+y)^2}{2}\frac{\partial^2 v}{\partial x \partial y} + \frac{x+y}{2}\frac{\partial v}{\partial y} - \frac{x+y}{2}\frac{\partial v}{\partial x} = v;$$

or

$$\frac{\partial^2 v}{\partial x \partial y} + \frac{1}{x+y}\frac{\partial v}{\partial y} - \frac{1}{x+y}\frac{\partial v}{\partial x} - \frac{2v}{(x+y)^2} = 0.$$

This equation may be written in the form,

$$\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} + \frac{v}{x+y}\right) - \frac{1}{x+y}\left(\frac{\partial v}{\partial x} + \frac{v}{x+y}\right) = 0.$$

Finally, let

(4) $$\frac{\partial v}{\partial x} + \frac{v}{x+y} = w.$$

Then

$$\frac{\partial w}{\partial y} - \frac{w}{x+y} = 0; \quad \text{or} \quad \frac{1}{x+y}\frac{\partial w}{\partial y} - \frac{w}{(x+y)^2} = 0,$$

[1] See Finkel's *Solution Book*, p. 319.